Symmetric Monotone Embedding of Traffic Flow Networks with First-In-First-Out Dynamics

Samuel Coogan * **Murat Arcak** **

* *Department of Electrical Engineering, University of California, Los Angeles (e-mail: scoogan@ucla.edu)*
** *Department of Electrical Engineering and Computer Sciences, University of California, Berkeley (e-mail: arcak@eecs.berkeley.edu)*

Abstract:
We study a flow network model for vehicular traffic that captures congestion effects at diverging junctions. Standard approaches which rely on monotonicity of the flow dynamics do not immediately apply to such *first-in-first-out* models. The network model nonetheless exhibits a *mixed monotonicity* property. Mixed monotonicity enables the original system to be embedded in a system of twice the dimension that is monotone and symmetric. The dynamics of the original system are recovered on a subspace of the embedding system, and we prove global asymptotic stability for a class of networks by considering convergence properties of the embedding system.

1. INTRODUCTION

We study the dynamical behavior of a model for vehicular traffic flow. The flow of vehicles from a *link* to downstream links is governed by a local flow *demand* as well as downstream *supply* of capacity available to accommodate incoming flow. Such an approach is well suited for modeling flow of vehicles on a freeway, Daganzo (1994, 1995), Gomes and Horowitz (2006), Gomes et al. (2008).

This paper builds on recent results in Gomes and Horowitz (2006), Gomes et al. (2008), Como et al. (2015), Coogan and Arcak (2014), Lovisari et al. (2014), and Coogan and Arcak (2015a) to analyze the dynamical behavior of transportation networks. In these prior works, the strongest results rely on the system dynamics being *monotone* whereby trajectories of the system preserve a partial ordering; see Hirsch (1985), Smith (1995). Yet, as detailed in Section 3.1, vehicular traffic networks with diverging junctions and fixed routing policies are not monotone, as noted in Coogan and Arcak (2014), Kurzhanskiy and Varaiya (2012). This is due to a first-in-first-out property where downstream congestion on one outgoing link blocks incoming flow to neighboring outgoing links, Munoz and Daganzo (2002).

In this paper, we first note that traffic dynamics possess a *mixed monotonicity* property which is much weaker than monotonicity. Despite the lack of monotonicity in the standard form of the dynamics, such systems can be embedded into a higher dimensional monotone system as in Enciso et al. (2006), Gouzé and Hadeler (1994), Kulenovic and Merino (2006), Smith (2008).

Next, we identify a certain class of *polytree* networks for which, using this embedding, we prove global asymptotic convergence to a unique equilibrium. In exchange for the generality offered by mixed monotonicity, we obtain only a sufficient condition that requires equilibria of the embedding system to be unique.

In Coogan and Arcak (2015b) and Coogan et al. (2015), we have noted that a similar discrete-time model of queue evolution in a network of signalized intersections is mixed monotone, but our focus was on finite state abstraction of the dynamics for automatic synthesis of control strategies rather than stability analysis. Furthermore, the model we consider for signalized intersections satisfies a sufficient condition for mixed monotonicity which is not generally satisfied by the freeway model considered here.

The remainder of the paper is organized as follows: In Section 2, we define the network model. In Section 3, we show that the dynamics are not monotone but are mixed monotone, which allows the system to be embedded in a higher dimensional monotone system. In Section 4, we apply these results to establish global stability for a class of networks. Concluding remarks are in provided Section 5.

2. TRAFFIC FLOW MODEL

2.1 Notation

All inequalities are interpreted elementwise, *e.g.*, for $x, y \in \mathbb{R}^n$, $x \leq y$ if and only if $x_i \leq y_i$ for $i = 1, \dots n$ where x_i, y_i denote the ith element of x, y. We denote the vector of all zeros by 0 when its dimension is clear from context. We denote the set of nonnegative real numbers by $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$.

2.2 Network Topology

We summarize the network model adopted here, which is based on the model developed in Coogan and Arcak (2014). A traffic network consists of a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{O})$ with *junctions* $\mathcal{V}$ and *ordinary links* $\mathcal{O}$ along

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with a set of *entry links* $\mathcal{R}$ which are entry points into the network. Let $\mathcal{L} \triangleq \mathcal{O} \cup \mathcal{R}$. Physically, a link represents a segment of roadway, and we assume $\mathcal{G}$ is a connected graph. Let $\sigma(\ell)$ and $\tau(\ell)$ denote the head and tail junction of link $\ell \in \mathcal{L}$, respectively, where we assume $\sigma(\ell) \neq \tau(\ell)$, *i.e.*, no self-loops. Traffic flows from $\tau(\ell)$ to $\sigma(\ell)$. By convention, $\tau(\ell) = \emptyset$ for all $\ell \in \mathcal{R}$.

For each $v \in \mathcal{V}$, we denote by $\mathcal{L}_v^{\text{in}} \subset \mathcal{L}$ the set of input links to node v and by $\mathcal{L}_v^{\text{out}} \subset \mathcal{L}$ the set of output links, *i.e.* $\mathcal{L}_v^{\text{in}} = \{\ell : \sigma(\ell) = v\}$ and $\mathcal{L}_v^{\text{out}} = \{\ell : \tau(\ell) = v\}$. We assume $\mathcal{L}_v^{\text{in}} \neq \emptyset$ for all $v \in \mathcal{V}$, thus the network flow starts at entry links. Furthermore, we assume $\mathcal{L}_{\sigma(\ell)}^{\text{out}} \neq \emptyset$ for all $\ell \in \mathcal{R}$, *i.e.* entry links always flow into at least one ordinary link downstream. If $|\mathcal{L}_v^{\text{in}}| > 1$, then v is a *merging junction*, and if $|\mathcal{L}_v^{\text{out}}| > 1$, then v is a *diverging junction*.

Define $\mathcal{V}^{\text{sink}} \triangleq \{v \in \mathcal{V} \mid \mathcal{L}_v^{\text{out}} = \emptyset\}$ to be the set of junctions that have no outgoing links and

$$\mathcal{L}^{\text{sink}} \triangleq \{\ell \in \mathcal{L} \mid \sigma(\ell) \in \mathcal{V}^{\text{sink}}\} \tag{1}$$

the corresponding set of input links to these junctions.

2.3 Link Supply and Demand

Each link $\ell \in \mathcal{L}$ has state $x_\ell(t) \in [0, \overline{x}_\ell]$ which is the mass of vehicles on link ℓ where $\overline{x}_\ell \in (0, \infty)$ is the maximum number of vehicles that link ℓ can accommodate. Let $\bar{x} = \{\bar{x}_\ell\}_{\ell \in \mathcal{L}}$. Furthermore, each link possesses a state-dependent *demand* function $\Phi_\ell^{\text{out}}(x_\ell)$ and *supply* function $\Phi_\ell^{\text{in}}(x_\ell)$ satisfying:

Assumption 1. For each $\ell \in \mathcal{L}$:

- The demand function $\Phi_\ell^{\text{out}} : [0, \overline{x}_\ell] \to \mathbb{R}_{\geq 0}$ is strictly increasing and Lipschitz continuous with $\Phi_\ell^{\text{out}}(0) = 0$.
- The supply function $\Phi_\ell^{\text{in}} : [0, \overline{x}_\ell] \to \mathbb{R}_{\geq 0}$ is strictly decreasing and Lipschitz continuous with $\Phi_\ell^{\text{in}}(\overline{x}_\ell) = 0$.

Assumption 1 implies that for each $\ell \in \mathcal{L}$, there exists unique x_ℓ^{crit} such that

$$\Phi_\ell^{\text{out}}(x_\ell^{\text{crit}}) = \Phi_\ell^{\text{in}}(x_\ell^{\text{crit}}) =: \Phi_\ell^{\text{crit}}. \tag{2}$$

The demand of a link is interpreted as the maximum outflow of the link, and the supply of a link is interpreted as the maximum inflow of the link.

2.4 Dynamic Model

At each junction $v \in \mathcal{V}$, there exists a collection of fixed *split* ratios $\{\beta_{\to\ell}\}_{\ell \in \mathcal{L}_v^{\text{out}}}$ with each $\beta_{\to\ell} > 0$ describing how incoming flow is split among outgoing links. Conservation of flow implies

$$\sum_{\ell \in \mathcal{L}_v^{\text{out}}} \beta_{\to\ell} \leq 1 \quad \forall v \in \mathcal{V}, \tag{3}$$

where strict inequality in (3) implies that a fraction of the flow is routed off the network via, *e.g.*, an unmodeled off-ramp.

Note that we associate a single split ratio with each output link rather than with each input/output link pair as in Coogan and Arcak (2014).Thus split ratios cannot differ for different incoming links, and, as we will see in Section 3, this leads to a mixed monotonicity property.

The flow dynamics of the network are as follows:

$$\dot{x}_\ell = f_\ell^{\text{in}}(x) - f_\ell^{\text{out}}(x) \qquad \forall \ell \in \mathcal{L} \tag{4}$$

$$=: F_\ell(x) \tag{5}$$

$$\alpha^v(x) \triangleq \min\left\{1, \min_{\ell \in \mathcal{L}_v^{\text{out}}} \left\{ \frac{\Phi_\ell^{\text{in}}(x_\ell)}{\beta_{\to\ell} \sum_{k \in \mathcal{L}_v^{\text{in}}} \Phi_k^{\text{out}}(x_k)} \right\}\right\} \forall v \in \mathcal{V} \tag{6}$$

$$f_\ell^{\text{out}}(x) = \alpha^{\sigma(\ell)}(x) \Phi_\ell^{\text{out}}(x_\ell) \tag{7}$$

$$f_\ell^{\text{in}}(x) = \begin{cases} \min\{d_\ell, \Phi_\ell^{\text{in}}(x_\ell)\} & \text{if } \ell \in \mathcal{R} \\ \beta_{\to\ell} \sum_{k \in \mathcal{L}_{\tau(\ell)}^{\text{in}}} f_k^{\text{out}}(x_k) & \text{if } \ell \in \mathcal{O}. \end{cases} \tag{8}$$

Above, $\alpha^v(x) \in [0, 1]$ defined in (6) is a factor that scales the outgoing flow of each input link at the junction v such that the incoming flow to each output link is less than its supply. Thus, the model (4)–(8) maximizes the flow through links while ensuring that the outgoing flow at each link does not exceed the link's demand and the incoming flow does not exceed the link's supply. For entry link $\ell \in \mathcal{R}$, the incoming flow is additionally restricted to not exceed the exogenous demand d_ℓ. The model further requires that, at each junction, the collection of outgoing flows for the input links is proportional to the collection of flow demand from the input links. This condition is referred to as *proportional-priority* in Kurzhanskiy and Varaiya (2010, 2012), and differs from the constant priority model employed in Coogan and Arcak (2015b), Coogan et al. (2015). From (4)–(8), we have forward invariance of the domain

$$\mathcal{X} \triangleq \prod_{\ell \in \mathcal{L}} [0, \overline{x}_\ell]. \tag{9}$$

The model (4)–(8) is modified from the model proposed in Coogan and Arcak (2014) so that entry links have finite capacity. This is reasonable for traffic networks where entry links correspond to onramps with finite storage capacity.

Define the routing matrices $R_\mathcal{O} \in \mathbb{R}^{\mathcal{O} \times \mathcal{O}}$ and $R_\mathcal{R} \in \mathbb{R}^{\mathcal{O} \times \mathcal{R}}$ elementwise as follows:

$$[R_\mathcal{O}]_{k\ell} = \begin{cases} \beta_{\to k} & \text{if } k \in \mathcal{L}_{\sigma(\ell)}^{\text{out}} \\ 0 & \text{otherwise} \end{cases} \quad \forall \ell, k \in \mathcal{O} \tag{10}$$

$$[R_\mathcal{R}]_{k\ell} = \begin{cases} \beta_{\to k} & \text{if } k \in \mathcal{L}_{\sigma(\ell)}^{\text{out}} \\ 0 & \text{otherwise} \end{cases} \quad \forall k \in \mathcal{O}, \forall \ell \in \mathcal{R}. \tag{11}$$

Assumption 2. The matrix $(I - R_\mathcal{O})$ is invertible.

Assumption 2 is equivalent to the assertion that eventually all vehicles will leave the network and is thus a natural assumption on the split ratios, Varaiya (2013). Let

$$P = (I - R_\mathcal{O})^{-1} R_\mathcal{R}, \tag{12}$$

that is, P describes how the flow from entry links is routed through the network. As $(I - R_\mathcal{O})^{-1} = I + R_\mathcal{O} + R_\mathcal{O}^2 + \ldots$, we have $P \geq 0$. Let

$$f_\ell^{\text{e}} = \begin{cases} d_\ell & \text{if } \ell \in \mathcal{R} \\ [Pd]_\ell & \text{if } \ell \in \mathcal{O}. \end{cases} \tag{13}$$

where $[Pd]_\ell$ is the ℓth entry of Pd.

Assumption 3. The input flow $d = \{d_\ell\}_{\ell \in \mathcal{R}}$ satisfies

$$f_\ell^{\text{e}} < \Phi_\ell^{\text{crit}} \quad \forall \ell \in \mathcal{L}. \tag{14}$$

Assumption 3 states that the network has adequate capacity to accommodate the input flow d, that is, d is *strictly feasible* according to Gomes et al. (2008). It follows from Assumption 3 that

$$x_\ell^{\mathrm{e}} \triangleq (\Phi_\ell^{\mathrm{out}})^{-1}(f_\ell^{\mathrm{e}}) < x_\ell^{\mathrm{crit}} \quad (15)$$

for all $\ell \in \mathcal{L}$ constitutes an equilibrium of the traffic network dynamics (4)–(8). Indeed, for this case, $\alpha^v(x^{\mathrm{e}}) = 1$ for all v where we define $x^{\mathrm{e}} = \{x_\ell^{\mathrm{e}}\}_{\ell \in \mathcal{L}}$, that is, the outgoing flow on every link is equal to demand. A key result of this paper is that this equilibrium is unique and globally asymptotically stable for a class of networks defined subsequently.

3. NONMONOTONE BEHAVIOR OF TRAFFIC NETWORKS

3.1 Lack of Monotonicity

Consider the system $\dot{x} = G(x)$, $x \in X \subseteq \mathbb{R}^n$ where X is forward invariant and has convex interior. Suppose $G(\cdot)$ is locally Lipschitz and satisfies

$$\frac{\partial G_i}{\partial x_j}(x) \geq 0 \quad \forall x \in X, \forall i \neq j \quad (16)$$

whenever the derivative exists. Then the system $\dot{x} = G(x)$ is order-preserving with respect to the positive orthant $\mathbb{R}^n_{\geq 0}$, that is,

$$x(0) \leq y(0) \text{ implies } x(t) \leq y(t) \quad \forall t \geq 0 \quad (17)$$

where $x(t)$, $y(t)$ are solutions of the system with initial conditions $x(0)$, $y(0)$. A dynamical system $\dot{x} = G(x)$ satisfying (16) is said to be *monotone with respect to the positive orthant*, or simply *monotone*, Hirsch (1985), Smith (1995).

Traffic flow networks with no diverging junctions are monotone, as has been noted and studied in Coogan and Arcak (2014), Gomes et al. (2008). However, networks with diverging junctions are not monotone. To see this, consider a diverging junction v and assume some link $\ell \in \mathcal{L}_v^{\mathrm{out}}$ is the unique minimizer in (6) for some x so that, for all y in a neighborhood of x,

$$\alpha^v(y) = \Phi_\ell^{\mathrm{in}}(y_\ell)\left(\beta_{\to\ell}\sum_{k\in\mathcal{L}_v^{\mathrm{in}}}\Phi_k^{\mathrm{out}}(y_k)\right)^{-1}. \quad (18)$$

It follows that, for all $k \in \mathcal{L}_v^{\mathrm{out}}$, $k \neq \ell$ with $\sigma(k) \neq \sigma(\ell)$,

$$\frac{\partial F_k}{\partial x_\ell}(x) = \frac{\partial f_k^{\mathrm{in}}}{\partial x_\ell}(x) = \frac{\partial}{\partial x_\ell}\left(\frac{\beta_{\to k}}{\beta_{\to\ell}}\Phi_\ell^{\mathrm{in}}(x_\ell)\right) < 0, \quad (19)$$

and thus the system is not monotone. We interpret (19) as follows: We assume the supply of downstream link ℓ is less than upstream demand due to congestion, and thus link ℓ inhibits flow through the junction. Therefore, an increase in the number of vehicles on link ℓ would worsen the congestion (decrease supply), and vehicles destined for link ℓ would further block flow to other outgoing links, causing a reduction in the incoming flow to these links. That is, the derivative of incoming flow to a downstream link $k \neq \ell$ with respect to link ℓ is nonzero and, in particular, is negative since $\Phi_\ell^{\mathrm{in}}(x_\ell)$ is a decreasing function. Thus, lack of monotonicity is indeed expected for traffic networks.

Remark 1. It is standard to generalize the condition (16) to partial orders with respect to arbitrary orthants as in Hirsch and Smith (2005), and one may wonder if the traffic dynamics are monotone with respect to some alternative orthant order. The answer is negative; indeed, the relationship (19) holds for any pair of output links, and for a junction with at least three output links, this implies that the system is not monotone with respect to any orthant via the graphical condition in, *e.g.*, (Angeli and Sontag, 2004, Proposition 2).

The phenomenon of downstream traffic blocking flow to other downstream links at a diverging junction is referred to as the *first-in-first-out (FIFO)* property, Daganzo (1995), Kurzhanskiy and Varaiya (2010), and it is a feature of traffic flow that has been observed even on wide freeways with many lanes, Cassidy et al. (2002), Munoz and Daganzo (2002). Some of the recent literature in dynamical flow models proposes alternative modeling choices for diverging junctions, *e.g.*, Como et al. (2015), Lovisari et al. (2014), which ensures that the resulting dynamics are monotone and therefore do not exhibit this FIFO property.

3.2 A Weaker Property: Mixed Monotonicity

The main result of this paper is that, while vehicular traffic networks are not monotone, they possess a weaker *mixed monotonicity* property. This property allows the traffic network dynamics to be embedded within a larger monotone system amenable to techniques for stability analysis of such systems.

We begin with a general characterization of mixed monotone systems which is a continuous-time analogue of the characterization in Smith (2008) and is closely related to recent results for nonmonotone interconnections of monotone systems, *e.g.*, Angeli et al. (2014).

Definition 1. (Mixed Monotone). The system $\dot{x} = G(x)$, $x \in X \subseteq \mathbb{R}^n$ where X has convex interior and G is locally Lipschitz is *mixed monotone* if there exists a locally Lipshitz continuous function $g(x, y)$ satisfying:

(1) $g(x, x) = G(x)$ for all $x \in X$

(2) $\dfrac{\partial g_i}{\partial x_j}(x, y) \geq 0$ for all $x, y \in X$ and all $i \neq j$ whenever the derivative exists

(3) $\dfrac{\partial g_i}{\partial y_j}(x, y) \leq 0$ for all $x, y \in X$ and all i, j whenever the derivative exists.

The function $g(x, y)$ is called a *decomposition function* for the system.

For a mixed monotone system with decomposition function $g(x, y)$, it follows that the symmetric system

$$\dot{x} = g(x, y) \quad (20)$$

$$\dot{y} = g(y, x) \quad (21)$$

is order-preserving with respect to the orthant $\mathbb{R}^n_{\geq 0} \times \mathbb{R}^n_{\leq 0}$. The following proposition, adapted from Enciso et al. (2006) and Smith (2006), makes use of this property to analyze the asymptotic behavior of

$$\dot{x} = g(x, x) = G(x), \quad (22)$$

whose trajectories coincide with (20)–(21) restricted to the diagonal $\{(x, x) \mid x \in X\}$.

Proposition 4. If there exist $x^0 \leq y^0$, $[x^0, y^0] \subseteq X$, such that

$$g(y^0, x^0) \leq 0 \leq g(x^0, y^0) \tag{23}$$

then $[x^0, y^0]$ is forward invariant for (22). Moreover, there exist $x^* \leq y^*$ for which $[x^*, y^*] \subseteq [x^0, y^0]$ and, letting $(x(t), y(t))$ denote the solution to (20)–(21) with initial condition (x^0, y^0), we have $(x(t), y(t)) \to (x^*, y^*)$, $g(x^*, y^*) = g(y^*, x^*) = 0$, and the ω-limit set of $[x^0, y^0]$ is nonempty and is contained within $[x^*, y^*]$.

Proof. Let $\leq_C$ be the order relation with respect to $C \triangleq \mathbb{R}^n_{\geq 0} \times \mathbb{R}^n_{\leq 0}$, that is, $(x, y) \leq_C (\tilde{x}, \tilde{y})$ if and only if $x \leq \tilde{x}$ and $\tilde{y} \leq y$. By hypothesis, $(x_0, y_0) \leq_C (y_0, x_0)$. Equation (23) implies

$$(g(y^0, x^0), g(x^0, y^0)) \leq_C (0, 0) \leq_C (g(x^0, y^0), g(y^0, x^0)). \tag{24}$$

Since the symmetric system (20)–(21) is order-preserving with respect to C, (24) implies that the set $\{(x, y) \mid (x_0, y_0) \leq_C (x, y) \leq_C (y_0, x_0)\}$ is forward invariant for (20)–(21) (Smith, 1995, Ch. 3, Prop. 2.1). Furthermore, the solution $(x(t), y(t))$ is monotonically increasing with respect to $\leq_C$. Symmetrically, $(y(t), x(t))$ is the solution to (20)–(21) with initial condition (y^0, x^0) and is monotonically decreasing. It follows that $(x(t), y(t)) \to (x^*, y^*)$ and both (x^*, y^*), (y^*, x^*) are equilibria of (20)–(21). Furthermore, $(x^*, y^*) \leq_C (y^*, x^*)$ since (20)–(21) is order preserving with respect to $\leq_C$, thus $x^* \leq y^*$.

Consider a trajectory $z(t)$ of (22) with initial condition $z^0 \in [x^0, y^0]$. This induces the corresponding trajectory $(z(t), z(t))$ of the symmetric system (20)–(21) with initial condition (z^0, z^0) satisfying $(x^0, y^0) \leq_C (z^0, z^0) \leq_C (y^0, x^0)$. It follows that $(x(t), y(t)) \leq_C (z(t), z(t)) \leq_C (y(t), x(t))$ for all time, that is, $x(t) \leq z(t)$ and $z(t) \leq y(t)$ for all time. We thus have $x^* \leq \lim_{t\to\infty} z(t) \leq y^*$. All claims of the Proposition then follow readily. □

Theorem 5. The traffic network model (4)–(8) is mixed monotone with decomposition function

$$g_\ell(x, y) = f^{\text{in}}_\ell(\xi^\ell(x, y)) - f^{\text{out}}_\ell(x). \tag{25}$$

Before providing the proof, we define notation that is used in the sequel. For all $\ell, k \in \mathcal{L}$, let

$$s_{\ell k} = \begin{cases} 1 & \text{if } \tau(k) = \tau(\ell) \text{ and } k \neq \ell \\ 0 & \text{else} \end{cases} \tag{26}$$

and for each $\ell \in \mathcal{L}$ and $x, y \in \mathbb{R}^{\mathcal{L}}$, let

$$\xi^\ell_k(x_k, y_k) = s_{\ell k} y_k + (1 - s_{\ell k}) x_k \quad \forall k \in \mathcal{L}, \tag{27}$$

$$\xi^\ell(x, y) = \{\xi^\ell_k(x_k, y_k)\}_{k\in\mathcal{L}}. \tag{28}$$

Proof. We first note that $F_\ell(x)$ is Lipschitz continuous for each $\ell \in \mathcal{L}$; in the following, statements involving derivatives are interpreted to hold whenever the derivative exists. For ease of notation, we interpret $f^{\text{in}}_\ell(x, y) = f^{\text{in}}_\ell(\xi^\ell(x, y))$. It holds trivially that $g_\ell(x, x) = F_\ell(x)$ for all $\ell \in \mathcal{L}$. We now show that

$$\frac{\partial f^{\text{out}}_\ell}{\partial x_k}(x) \leq 0 \quad \forall x \in \mathcal{X}, \forall \ell \neq k \tag{29}$$

$$\frac{\partial f^{\text{in}}_\ell}{\partial x_k}(x, y) \geq 0 \quad \forall x, y \in \mathcal{X}, \forall \ell \neq k \tag{30}$$

$$\frac{\partial f^{\text{in}}_\ell}{\partial y_k}(x, y) \leq 0 \quad \forall x, y \in \mathcal{X}, \forall k \tag{31}$$

which implies that $g_\ell(x, y)$ is indeed a valid decomposition function for the system and the system is mixed monotone, completing the proof.

To this end, first consider $k \in \mathcal{L}^{\text{out}}_{\sigma(\ell)}$. We have

$$\frac{\partial f^{\text{out}}_\ell}{\partial x_k}(x) = \frac{\partial \alpha^{\sigma(\ell)}}{\partial x_k}(x)\Phi^{\text{out}}_\ell(x_\ell) \tag{32}$$

$$\in \left\{0, \frac{\Phi^{\text{out}}_\ell(x_\ell)}{\sum_{j\in\mathcal{L}^{\text{in}}_{\sigma(\ell)}} \beta_{\to k}\Phi^{\text{out}}_j(x_j)} \frac{d\Phi^{\text{in}}_k}{dx_k}(x_k)\right\} \leq 0. \tag{33}$$

If $k \in \mathcal{L}^{\text{in}}_{\sigma(\ell)}$, then (32) still holds and whenever $\partial\alpha^{\sigma(\ell)}/\partial x_k \neq 0$, there exists $m \in \mathcal{L}^{\text{out}}_{\sigma(\ell)}$ such that

$$\alpha^{\sigma(\ell)}(x) = \left(\sum_{j\in\mathcal{L}^{\text{in}}_{\sigma(\ell)}} \beta_{\to m}\Phi^{\text{out}}_j(x_j)\right)^{-1} \Phi^{\text{in}}_m(x_m) \tag{34}$$

$$\frac{\partial \alpha^{\sigma(\ell)}}{\partial x_k}(x) = -\frac{\frac{d\Phi^{\text{out}}_k}{dx_k}(x_k)\Phi^{\text{in}}_m(x_m)}{\beta_{\to m}\left(\sum_{j\in\mathcal{L}^{\text{in}}_{\sigma(\ell)}} \Phi^{\text{out}}_j(x_j)\right)^2} \leq 0, \tag{35}$$

and thus (29) holds. Next, we have

$$\frac{\partial f^{\text{in}}_\ell}{\partial x_k}(x, y) = \begin{cases} \beta_{\to\ell} \sum_{j\in\mathcal{L}^{\text{in}}_{\tau(\ell)}} \frac{\partial f^{\text{out}}_j}{\partial x_k}(\xi^\ell(x, y)) & \text{if } k \in \mathcal{L}^{\text{in}}_{\tau(\ell)} \\ 0 & \text{else} \end{cases} \tag{36}$$

by (26)–(28). For $k \in \mathcal{L}^{\text{in}}_{\tau(\ell)}$, we have

$$\sum_{j\in\mathcal{L}^{\text{in}}_{\tau(\ell)}} \frac{\partial f^{\text{out}}_j}{\partial x_k}(\xi^\ell(x, y)) = \frac{\partial \alpha^{\tau(\ell)}}{\partial x_k}(\xi^\ell(x, y)) \sum_{j\in\mathcal{L}^{\text{in}}_{\tau(\ell)}} \Phi^{\text{out}}_j(x_j) + \alpha^{\tau(\ell)}(\xi^\ell(x, y))\frac{d\Phi^{\text{out}}_k}{dx_k}(x_k). \tag{37}$$

If $\alpha^{\tau(\ell)} \neq 1$ on some neighborhood of $\xi^\ell(x, y)$, then there exists $m \in \mathcal{L}^{\text{out}}_{\tau(\ell)}$ such that

$$\alpha^{\tau(\ell)}(\xi^\ell(x, y)) = \left(\sum_{j\in\mathcal{L}^{\text{in}}_{\tau(\ell)}} \beta_{\to m}\Phi^{\text{out}}_j(x_j)\right)^{-1} \Phi^{\text{in}}_m(\xi^\ell_m(x, y)), \tag{38}$$

$$\frac{\partial \alpha^{\tau(\ell)}}{\partial x_k}(\xi^\ell(x, y)) = -\frac{\frac{d\Phi^{\text{out}}_k}{dx_k}(x_k)\Phi^{\text{in}}_m(\xi^\ell_m(x, y))}{\beta_{\to m}\left(\sum_{j\in\mathcal{L}^{\text{in}}_{\tau(\ell)}} \Phi^{\text{out}}_j(x_j)\right)^2}. \tag{39}$$

Then (37) evaluates to zero for this case. Therefore,

$$\frac{\partial f^{\text{in}}_\ell}{\partial x_k}(\xi^\ell(x, y)) \in \left\{0, \beta_{\to\ell}\frac{d\Phi^{\text{out}}_k}{dx_k}(x_k)\right\} \geq 0, \tag{40}$$

and thus (30) holds. Finally, we have

$$\frac{\partial f^{\text{in}}_\ell}{\partial y_k}(x, y) = \begin{cases} \beta_{\to\ell}\frac{\partial \alpha^{\tau(\ell)}}{\partial x_k}(\xi^\ell(x, y)) \sum_{j\in\mathcal{L}^{\text{in}}_{\tau(\ell)}} \Phi^{\text{out}}_j(x_j) & \text{if } \tau(k) = \tau(\ell) \text{ and } k \neq \ell \\ 0 & \text{else.} \end{cases} \tag{41}$$

If $\frac{\partial \alpha^{\tau(\ell)}}{\partial x_k}(\xi^\ell(x, y)) \neq 0$ for some $k \neq \ell$ with $\tau(k) = \tau(\ell)$, then it must be that

$$\frac{\partial \alpha^{\tau(\ell)}}{\partial x_k}(\xi^\ell(x, y)) = \left(\sum_{j\in\mathcal{L}^{\text{in}}_{\tau(\ell)}} \beta_{\to k}\Phi^{\text{out}}_j(x_j)\right)^{-1} \frac{d\Phi^{\text{in}}_k}{dx_k}(y_k). \tag{42}$$

We conclude that (31) holds because

$$\frac{\partial f_\ell^{\text{in}}}{\partial y_k}(x,y) \in \left\{0, \frac{\beta_{\to\ell}}{\beta_{\to k}}\frac{d\Phi_k^{\text{in}}}{dx_k}(y_k)\right\} \leq 0. \tag{43}$$

□

4. GLOBAL STABILITY OF POLYTREE NETWORKS

We now use Theorem 5 and the order-preserving properties of (20)–(21) to prove global stability of a particular class of traffic networks.

Definition 2. The connected graph $\mathcal{G}$ is said to be a *polytree* graph if the underlying undirected graph is acyclic.

The "underlying undirected graph" is the undirected graph that results from replacing each directed edge with an undirected edge between the same two nodes.

The class of networks that constitutes polytree graphs is somewhat restrictive, as it does not allow cycles or multiple paths between two locations. However, polytrees still encompass a large class of relevant networks, such as a stretch of freeway with onramps and offramps, or a portion of a freeway network leading into (resp. out from) a large metropolitan area, which is useful for modeling the morning (resp. evening) commute patterns in the area. Furthermore, it is not difficult to construct a counterexample to demonstrate that the polytree assumption for the network topology is necessary for the results presented here.

Theorem 6. The equilibrium x^{e} identified in (15) is globally asymptotically stable for polytree networks.

Proof. We have

$$0 \leq g_\ell(0,\bar{x}) = \begin{cases} d_\ell & \text{if } \ell \in \mathcal{R} \\ 0 & \text{if } \ell \in \mathcal{O} \end{cases} \tag{44}$$

$$0 \geq g_\ell(\bar{x},0) = \begin{cases} 0 & \text{if } \ell \in \mathcal{L}\backslash\mathcal{L}^{\text{sink}} \\ -\Phi_\ell^{\text{out}}(\bar{x}_\ell) & \text{if } \ell \in \mathcal{L}^{\text{sink}}. \end{cases} \tag{45}$$

Let $(x(t),y(t))$ be the solution to (20)–(21) with initial condition $(0,\bar{x})$. Taking $(x^0,y^0)=(0,\bar{x})$, it follows from the proof of Proposition 4 that $(x(t),y(t)) \to (x^*,y^*)$ for some (x^*,y^*) an equilibrium of (20)–(21). Because $(x^{\text{e}},x^{\text{e}})$ is trivially also an equilibrium of (20)–(21) and since $x^0 \leq x^{\text{e}} \leq y^0$, we have that

$$x^* \leq x^{\text{e}} \leq y^* \tag{46}$$

by the order preserving property of (20)–(21). By symmetry, (y^*,x^*) is also an equilibrium of (20)–(21). That is, $f_\ell^{\text{in}}(\xi^\ell(x^*,y^*)) = f_\ell^{\text{out}}(y^*)$ for all $\ell \in \mathcal{L}$, and $f_\ell^{\text{in}}(\xi^\ell(y^*,x^*)) = f_\ell^{\text{out}}(x^*)$ for all $\ell \in \mathcal{L}$. We now show that $x^*=y^*=x^{\text{e}}$.

Suppose $y^* \neq x^{\text{e}}$. Recalling that $y^* \geq x^{\text{e}}$, this implies there exists $\ell \in \mathcal{L}$ such that $y_\ell^* > x_\ell^{\text{e}}$. By acyclicity of polytree networks, we assume, without loss of generality, that there does not exists $k \in \mathcal{L}_{\sigma(\ell)}^{\text{out}}$ such that $y_k^* > x_k^{\text{e}}$ (otherwise we could choose link k instead of ℓ). This implies that $\Phi_k^{\text{in}}(y_k^*) = \Phi_k^{\text{in}}(x_k^{\text{e}})$ for all $k \in \mathcal{L}_{\sigma(\ell)}^{\text{out}}$. Without loss of generality, we further assume that $f_\ell^{\text{out}}(y^*) > f_\ell^{\text{e}}$. Indeed, if this were not the case, then there must exist some downstream link with inadequate supply since $\Phi_\ell^{\text{out}}(y_\ell^*) > \Phi_\ell^{\text{out}}(x_\ell^{\text{e}}) = f_\ell^{\text{e}}$. That is, there exists $k \in \mathcal{L}_{\sigma(\ell)}^{\text{out}}$, for which $y_k^* = x_k^{\text{e}}$, such that

$$\sum_{j\in\mathcal{L}_{\sigma(\ell)}^{\text{in}}} f_j^{\text{out}}(y^*) = (1/\beta_{\to k}) f_k^{\text{in}}(y_k^*) > \sum_{j\in\mathcal{L}_{\sigma(\ell)}^{\text{in}}} f_j^{\text{e}}, \tag{47}$$

for which there must exist some $j \in \mathcal{L}_{\sigma(\ell)}^{\text{in}}$, $j \neq \ell$ with $f_j^{\text{out}}(y^*) > f_j^{\text{e}}$, and we could choose j instead of ℓ.

We thus have $f_\ell^{\text{in}}(\xi^\ell(y^*,x^*)) = f_\ell^{\text{out}}(y^*) > f_\ell^{\text{e}}$. Define $\ell_0 \triangleq \ell$ and, starting from ℓ_0, choose inductively $\ell_1,\ell_2,\dots,\ell_n$ to satisfy $\ell_i \in \mathcal{L}_{\tau(\ell_{i-1})}^{\text{in}}$ for all i (that is, link ℓ_i is upstream of link ℓ_{i-1}) such that $f_{\ell_i}^{\text{out}}(y^*) > f_{\ell_i}^{\text{e}}$, until no additional upstream link satisfying this condition exists. Note it is possible that $\ell_n = \ell_0 = \ell$. Thus,

$$f_{\ell_n}^{\text{in}}(\xi^{\ell_n}(y^*,x^*)) = f_{\ell_n}^{\text{out}}(y^*) > f_{\ell_n}^{\text{e}}. \tag{48}$$

Since $f_k^{\text{e}} = d_k$ for all $k \in \mathcal{R}$ by Assumption 3 and $f_k^{\text{in}}(\xi^k(y^*,x^*)) \leq d_k$ for all $k \in \mathcal{R}$, we have $\ell_n \neq \mathcal{R}$. Thus $f_j^{\text{out}}(y^*) \leq f_j^{\text{e}}$ for all $j \in \mathcal{L}_{\tau(\ell_n)}^{\text{in}}$, which implies $\beta_{\to\ell_n}\sum_{j\in\mathcal{L}_{\tau(\ell)}^{\text{in}}} f_j^{\text{out}}(y^*) = f_{\ell_n}^{\text{in}}(y^*) \leq f_{\ell_n}^{\text{e}}$. With (48), this implies that $f_{\ell_n}^{\text{in}}(\xi^{\ell_n}(y^*,x^*)) > f_{\ell_n}^{\text{in}}(y^*)$. Thus there must exist k with $\tau(k)=\tau(\ell_n)$ and $k \neq \ell_n$ such that the supply of link k at state y^* limits the outflow of the upstream links, that is, link k is such that $\beta_{\to k}\sum_{j\in\mathcal{L}_{\tau(k)}^{\text{in}}} f_j^{\text{out}}(y^*) = \Phi_k^{\text{in}}(y_k^*)$ for which $\Phi_k^{\text{in}}(y_k^*) \leq f_k^{\text{e}}$.

Define $k_0 = k$ and construct another sequence $k_1,k_2,\dots,k_n$ satisfying $k_i \in \mathcal{L}_{\sigma(k_{i-1})}^{\text{out}}$ for all i (that is, link k_i is downstream of link k_{i-1}) such that $\Phi_{k_i}^{\text{in}}(y_{k_i}^*) \leq f_{k_i}^{\text{e}}$, until no additional downstream link satisfying this condition exists. Note it is possible that $k_n = k_0 = k$. It thus holds that $\Phi_{k_n}^{\text{in}}(y_{k_n}^*) \leq f_{k_n}^{\text{e}}$ and $\Phi_j^{\text{in}}(y_j^*) > f_j^{\text{e}}$ for all $j \in \mathcal{L}_{\sigma(k_n)}^{\text{out}}$. Recall $f_{k_n}^{\text{out}}(y^*) \leq \Phi_{k_n}^{\text{in}}(y_{k_n}^*) < \Phi_{k_n}^{\text{out}}(y_{k_n}^*)$, where the second inequality follows because $\Phi_{k_n}^{\text{in}}(y_{k_n}^*) \leq f_{k_n}^{\text{e}}$ and thus $y_{k_n}^* > x_{k_n}^{\text{crit}}$. Therefore there exists $j \in \mathcal{L}_{\sigma(k_n)}^{\text{out}}$ such that $\beta_{\to j}\sum_{m\in\mathcal{L}_{\sigma(k_n)}^{\text{in}}} f_m^{\text{out}}(y^*) = \Phi_j^{\text{in}}(y_j^*)$, but then there must exist $m \in \mathcal{L}_{\sigma(k_n)}^{\text{in}}$ such that $f_m^{\text{out}}(y^*) > f_m^{\text{e}}$ since $\Phi_j^{\text{in}}(y_j^*) > f_j^{\text{e}}$, for which $y_m^* > x_m^{\text{e}}$. However, $m \neq \ell_i$ for any $\ell_i \in \{\ell_0,\dots,\ell_n\}$ chosen previously, as this would imply the graph is not a polytree. Taking $\ell = m$, we could begin the process again, continuing indefinitely; since the graph is finite, we arrive at a contradiction, and thus we must have $y^* = x^{\text{e}}$.

Now suppose $x^* \neq x^{\text{e}}$, that is, there exists ℓ such that $x_\ell^* < x_\ell^{\text{e}}$. Without loss of generality, assume there does not exist $k \in \mathcal{L}_{\tau(\ell)}^{\text{in}}$ such that $x_k^* < x_k^{\text{e}}$. This implies

$$f_\ell^{\text{out}}(x^*) \leq \Phi_\ell^{\text{out}}(x_\ell^*) < f_\ell^{\text{e}}. \tag{49}$$

Since $f_\ell^{\text{in}}(\xi^\ell(x^*,y^*)) = f_\ell^{\text{out}}(x^*)$ and

$$\beta_{\to\ell}\sum_{k\in\mathcal{L}_{\tau(\ell)}^{\text{in}}} \Phi_k^{\text{out}}(x_k^*) = \beta_{\to\ell}\sum_{k\in\mathcal{L}_{\tau(\ell)}^{\text{in}}} \Phi_k^{\text{out}}(x_k^{\text{e}}) = f_\ell^{\text{e}}, \tag{50}$$

there must exist $j \in \mathcal{L}_{\tau(\ell)}^{\text{out}}$, $j \neq \ell$, such that $f_\ell^{\text{in}}(\xi^\ell(x^*,y^*)) = (\beta_{\to\ell}/\beta_{\to j})\Phi_j^{\text{in}}(y_j^*)$, for which we must have $y_j^* > x_j^{\text{e}}$, a contradiction since we have shown $y^* = x^{\text{e}}$. As $x^* = y^* = x^{\text{e}}$, we apply Proposition 4 and conclude that the equilibrium x^{e} is globally attractive.

Finally, suppose the links are indexed $1,\dots,|\mathcal{L}|$ such that the index of link ℓ is less than the index of each $k \in \mathcal{L}_{\sigma(\ell)}^{\text{out}}$ (such an indexing is always possible for polytree graphs). Then the Jacobian evaluated at the equilibrium, $(\partial F/\partial x)(x^{\mathrm{e}})$, is lower triangular with respect to this indexing since $f_\ell^{\mathrm{out}}(x^{\mathrm{e}}) = \Phi_\ell^{\mathrm{out}}(x_\ell^{\mathrm{e}})$ for all $\ell \in \mathcal{L}$. Additionally, the Jacobian contains strictly negative entries along the diagonal since $\Phi_\ell^{\mathrm{out}}(\cdot)$ is strictly increasing, and it is therefore Hurwitz. Thus the equilibrium is locally asymptotically stable by, *e.g.*, (Khalil, 2002, Thereom 4.7) and therefore globally asymptotically stable since it is also globally attractive. □

5. CONCLUSIONS

We have characterized a *mixed monotonicity* property exhibited by traffic flow networks. This property has potential to be a key tool in overcoming hurdles towards a full understanding of the dynamics of various physically-motivated flow networks, particularly related to transportation systems. We have shown that mixed monotone systems are able to be embedded in a higher dimensional system with twice the statespace dimension. Within this embedding system, the dynamics are monotone.

By studying the behavior of the embedding system, we established convergence properties of the original mixed monotone system. By exploiting this result, we proved global stability of traffic flow networks for a class of networks in which the graph topology is a polytree, that is, the undirected underlying graph does not contain any cycles. Future work will seek to extend these analysis techniques to broader classes of systems.

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