A Numerically Stable Dynamic Mode Decomposition Algorithm for Nearly Defective Systems

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***Abstract*— The potential for numerical instabilities of Dynamic Mode Decomposition (DMD), which assumes the completeness of the eigenspace is discussed for cases where the underlying system is defective or nearly defective. A numerically stable approach based on Schur decomposition is presented. The proposed method complements the DMD for cases where eigendecomposition is ill-conditioned. Both mathematical analysis and the results of numerical experiments are presented.**

***Index Terms*— computational methods, numerical algorithms, identification, identification for control, large-scale systems**

I. INTRODUCTION

FOR systems of high complexity and dimensionality, building models empirically rather than analytically is much more practical, especially when high definition data is readily available. Dynamic Mode Decomposition (DMD) [1], [2] is one of the most notable data-driven methods, which enables the prediction of the system's future behavior from current measurements. Recently, it was shown that a simple extension may be done to determine the effect of actuation on dynamical systems through DMD with control (DMDc) [3]. Therefore, DMD, which has strong connections to model reduction [4], Eigen Realization Algorithm (ERA) [5] and Observer Kalman Identification (ORKID) [6] has recently attracted much interest in the control community lending itself as a reduced order modeling method for high dimensional systems in the context of identification for control [7]–[10] . The result in [11], which showed that DMD approximates the Koopman operator, an infinite dimensional linear operator that can be used to represent nonlinear systems, ignited even more research interest in DMD. Since then, additional advancements have been made between DMD and Koopman theory in parallel [12]–[14]. However, DMD algorithms are based on eigendecomposition, which assumes the completeness of the eigenspace associated with the underlying system. Perhaps this assumption is justified in the sense that the set of matrices with a complete eigenspace forms a dense subset of the set of square matrices of a given order. Although the chances of encountering a truly defective systems may be slim, a system can get arbitrarily close to defectiveness. If the underlying system is nearly defective, such characteristic will manifest as clustered eigenvalues or complex conjugate eigenvalue pairs with very small imaginary parts. These scenarios are frequently encountered in many applications and have created much research interests [15]–[18]. This paper evaluates the numerical stability of DMD algorithms for nearly defective cases and proposes a Schur decomposition based approach, whose numerical stability remains robust for nearly defective systems. Our method, which allows numerically stable and accurate computations of solutions for defective and nearly defective systems is extremely useful for applications such as eigensensitivity analysis of structural optimal design and structural damage detection in structural engineering [15]. For example, sensitivity-based finite element model updating in such problems requires numerically stable solutions to determine their sensitivity to variations in system parameters and defective systems present a particular challenge in this context. Also, defective systems arise in many mechanical systems with general viscous damping [19] and a classic example is the torsional vibration of marine propulsion systems [20]. Our Schur-decomposition based approach, unlike current DMD, is highly applicable to these engineering problems. Since diagonalizability is not assumed, computations in our method involve an upper triangular matrix obtained from Schur decomposition as opposed to a diagonal matrix obtained from the eigendecompositon in DMD. Our approach is not to replace prior DMD algorithms but rather to complement them in cases where the eigendecomposition is ill-conditioned. In terms of computational ease, the next best structure to a diagonal matrix is, perhaps, a block diagonal *Jordan* form, which exists for any square matrix. However, computation of the *Jordan* decomposition is numerically unstable as will be demonstrated in the numerical examples of Section IV. Utilizing Schur decomposition yields

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the best possible condition numbers for the transformation matrix due to the unitary factor in the *Schur* form. We dub our method Schur-based DMD simply because it is closely related to DMD methods albeit not involving the computation of eigenmodes.

The paper is organized as follows. The mathematical background and definitions are given in Section II and the main results of the paper are presented in Section III. First, a theorem that evaluates the numerical stability of the DMD method is presented. Subsequently, a theorem which provides a realistic upper bound for the condition number of the transformation matrix in our method is presented. The numerical examples are provided in Section IV. The conclusions of the paper are summarized in Section V.

II. Mathematical Preliminaries

Definition 1: Matrix 2-norm
The matrix 2-norm of $A \in \mathbb{C}^{m\times n}$ is defined as $||A|| = \max_{||x||=1} ||Ax||_2$.

Remark: It can be easily verified that the matrix 2-norm is the largest singular value of A. It is also worth noting that the matrix 2-norm satisfies the sub-multiplicative property, i.e, $||AB|| \leq ||A||\ ||B||$. Unless otherwise indicated $||.||$ will denote the matrix 2-norm throughout the paper.

Definition 2: Condition number
The condition number of a full rank matrix $A \in \mathbb{C}^{m\times n}$ is defined as $\kappa(A) = ||A||\ ||A^+||$, where A^+ denotes the Moore-Penrose pseudo-inverse. If A is rank deficient, $\kappa(A)$ is defined to be $+\infty$.

Remark: The smallest possible condition number for any full rank matrix is unity.

The following theorem states that the reciprocal of the condition number of a full rank matrix is the relative distance to the nearest rank deficient matrix. The proof for the square matrices can be found in [21]. With minor modifications, the argument in [21] can be extended to rectangular matrices. Even more general definition of the condition number and its properties for 1, 2 and ∞-norm can be found in [22] but this paper only concerns with the 2-norm.

Theorem 1: [21], [22] *If $A \in \mathbb{C}^{m\times n}$ is a full rank matrix,*

$$\frac{1}{\kappa(A)} = \min_B \left\{ \frac{||A-B||}{||A||} : \text{rank}(B) < \text{rank}(A) \right\}.$$

Schur Decomposition is one of the most popular matrix decompositions, which can be computed by numerically stable algorithms and it is usually used as an intermediate step in computing eigendecomposition. The following theorem states the existence of the Schur form of a square matrix.

Theorem 2: [23] *If $A \in \mathbb{C}^{m\times m}$, then, $A = Q^*RQ$, where Q is unitary, R is upper triangular and Q^* denotes the conjugate transpose of Q.*

A. Exact Dynamic Mode Decomposition (eDMD)

The eDMD algorithm proposed by [2], usually used for high dimensional systems provides the solution for $x_{k+1} = Ax_k$ through a locally linear approximation using large matrices X and X' constructed from the snapshots of the data $X' = AX$. Then, the best-fit matrix for $A \in \mathbb{C}^{m\times m}$ is given by $A \approx X'X^+$. However, DMD circumvents having to directly compute the eigenvectors and eigenvalues of a large A matrix in the following manner. First, low rank structure of the data matrix X is extracted by singular value decomposition so that rank r approximation is $X \approx U\Sigma V^*$, where $U \in \mathbb{C}^{m\times r}$, $\Sigma \in \mathbb{C}^{r\times r}$ and $V \in \mathbb{C}^{r\times n}$. Then A is projected on to proper orthogonal decomposition (POD) modes of U yielding an $r\times r$ matrix, $\tilde{A} = U^*AU$. Then the eigendecomposition is performed for $\tilde{A}$, which yields, $\tilde{A}W = W\Lambda$, where the columns of W are the eigenvectors of $\tilde{A}$ and Λ is the diagonal matrix containing the eigenvalues of $\tilde{A}$ on the diagonal. Now it can be easily verified that $\Phi = X'V\Sigma^{-1}W$ satisfies $A\Phi = \Phi\Lambda$. Therefore, the approximate solution in discrete time is obtained as $x(k) \approx \Phi\Lambda^k\Phi^+x_0$, where Φ^+ is the Moore-Penrose pseudo-inverse and x_0 is the initial condition.

B. DMD for Control (DMDc)

In [3] it is shown that DMD can be extended to allow control inputs. Instead of $X' = AX$, it considers $X' = AX + B\Theta$, where Θ is constructed from time snapshots of the input. Denoting $G = \begin{bmatrix} A & B \end{bmatrix}$ and $\Omega = \begin{bmatrix} X \\ U \end{bmatrix}$, if $\Omega \approx U\Sigma V^*$ with rank p truncation, then A and B can be extracted as $A \approx X'V\Sigma^{-1}U_1^*$ and $B \approx X'V\Sigma^{-1}U_2^*$ respectively, where $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$. Suppose $X' = \bar{U}\bar{\Sigma}\bar{V}^*$ with rank r truncation, and project A and B onto a lower dimensional space using $\bar{U}$, i.e, $\tilde{A} = \bar{U}^*A\bar{U}$ and $\tilde{B} = \bar{U}^*B$. Then the dynamic modes of A are computed as $\Phi = X'\bar{V}\bar{\Sigma}U_1^*\bar{U}W$, where W is computed from $\tilde{A}W = W\Lambda$. Therefore, DMD is of much interest for control system research. Moreover, since DMD can be used to approximate the nonlinear systems through their Koopman operator representation, the method is of major interest for nonlinear system identification and control. This paper is concerned with the numerical stability of DMD and the ideas presented in this paper equally apply to both autonomous case and non-autonomous case. For simplicity, the arguments will be made in the autonomous setting without loss of generality.

III. Main Results

The following result is useful to analyze the numerical stability of DMD.

Theorem 3: *Let A and B be full rank matrices where the product AB is well defined. Then,*

$$\kappa(AB) \geq \frac{||AB||}{||A||}\frac{1}{\sigma_{min}(B)},$$

where $\sigma_{min}(B)$ denotes the smallest singular value of B.

Proof: Let $\bar{C}$ be the minimizer such that

$$\bar{C} = \arg\min_{C} \left\{ \frac{||B-C||}{||B||},\ \text{rank}(C) < \ \text{rank}(B) \right\}.$$

Then,

$$\begin{aligned} \frac{1}{\kappa(B)} &= \frac{||B-\bar{C}||}{||B||} \\ \frac{||A||}{\kappa(B)} &\geq \frac{||AB - A\bar{C}||}{||B||}. \end{aligned} \tag{1}$$

On the other hand, since $A\bar{C}$ is rank deficient,

$$\frac{1}{\kappa(AB)} \leq \frac{||AB - A\bar{C}||}{||AB||}. \tag{2}$$

Thus, from (1) and (2)

$$\begin{aligned} \frac{||A||}{\kappa(B)} &\geq \frac{||AB||}{\kappa(AB)||B||} \\ \kappa(AB) &\geq \frac{||AB||}{||A||}\frac{1}{\sigma_{min}(B)}. \end{aligned}$$

Thus, the theorem is proved.

In the DMD methods in the literature [2] the eigenvectors of the underlying systems are computed from columns of $\Phi = X'V\Sigma^{-1}W$. In cases where $X'V$ and W have full rank, by Theorem 3,

$$\kappa(\Phi) \geq \frac{||\Phi||}{||X'V\Sigma^{-1}||}\frac{1}{\sigma_{min}(W)}.$$

Notice that for any nonzero W, $||\Phi||$ is nonzero since $X'V\Sigma^{-1}$ has a trivial null space, and $\kappa(\Phi)$ can potentially be extremely high for small $\sigma_{min}(W)$. In all of our numerical experiments small $\sigma_{min}(W)$ values always produce large $\kappa(\Phi)$. See Table I. Therefore, care should be taken when applying such DMD algorithms on nearly defective systems. If the system is, in fact, truly defective, Φ will have an infinite condition number.

A. Schur-based DMD

In this paper we propose a Schur decomposition-based alternative in which the diagonalizability of the underlying matrix A is not assumed and the solution is computed using Schur decompositon instead of eigendecomposition. Without the assumption of diagonalizability the diagonal matrix of eigenvalues is replaced by an upper triangular matrix. Even though the convenience of the diagonalization is lost, the solutions obtained here are more numerically stable due to the unitary matrices of the Schur decomposition. Our method is outlined below. Proceed using the same steps as eDMD until $\tilde{A}$ is constructed. Then, perform Schur decomposition on $\tilde{A}$ so that

$$\tilde{A} = Q^*RQ,$$

where Q is a unitary matrix and R is an upper triangular matrix. Then construct the transformation matrix $\Phi_{Schur} = X'V\Sigma^{-1}Q$. It can be easily verified that

$$A\Phi_{Schur} = \Phi_{Schur}R.$$

Therefore, the discrete solution can be approximated as

$$x(k) \approx \Phi_{Schur}R^k\Phi_{Schur}^{+}x_0.$$

This method is not necessarily meant to replace the DMD but rather to complement it as follows. In the process of performing DMD if the condition number of the eigenvector matrix W is over the tolerable value, the control should be routed to a subroutine with our method. Schur decomposition is usually an intermediate step in computing the eigendecomposition. Therefore, once the eigendecomposition has been done Schur vectors are readily available. We propose utilizing our method as below.

1) Perform low rank singular value decomposition of X, $X \approx U\Sigma V^*$.
2) Compute $\tilde{A} = U^*AU = U^*X'X^+U$.
3) Compute eigendecomposition of $\tilde{A}$.
$$\tilde{A}W = \Lambda W.$$
4) Check the condition number of W. If less than the tolerance, proceed with DMD. Otherwise go to the next step in this procedure.
5) Compute the transformation matrix Φ_{Schur} using the Schur vector matrix Q.
$$\Phi_{Schur} = X'V\Sigma^{-1}Q.$$

The following theorem provides an upper bound for the condition number of the transformation matrix Φ_{Schur}.

Theorem 4: Let $\Phi_{Schur} = X'V\Sigma^{-1}Q$, where Σ is invertible and Q is unitary. Then,

$$\kappa(\Phi_{Schur}) \leq \kappa(X'V\Sigma^{-1}).$$

If $X'V$ has full column rank,

$$\kappa(\Phi_{Schur}) \leq \kappa(X'V)\kappa(\Sigma).$$

Proof: Since Q is unitary, the pseudo-inverse can be distributed over the product as

$$(X'V\Sigma^{-1}Q)^+ = Q^*(X'V\Sigma^{-1})^+.$$

Now, observe,

$$\begin{aligned} \kappa(\Phi_{Schur}) &= ||X'V\Sigma^{-1}Q||\ ||(X'V\Sigma^{-1}Q)^+|| && \text{(3a)} \\ &= ||X'V\Sigma^{-1}Q||\ ||Q^*(X'V\Sigma^{-1})^+|| && \text{(3b)} \\ &\leq ||X'V\Sigma^{-1}||\ ||Q||\ ||Q^*||\ ||(X'V\Sigma^{-1})^+|| && \text{(3c)} \\ &= \kappa(X'V\Sigma^{-1}). && \text{(3d)} \end{aligned}$$

The inequality in the third step was obtained by applying the submultiplicative property of the matrix 2-norm. The last equality follows due to the fact that the condition number of a unitary matrix is 1. Note that Σ^{-1} has full row rank. Thus, if $X'V$ has full column rank, the pseudo-inverse can be distributed over the product as

$$(X'V\Sigma^{-1})^+ = \Sigma(X'V)^+.$$

Thus, using the arguments similar to (3) it can be easily verified that

$$\kappa(\Phi_{Schur}) \leq \kappa(X'V)\ \kappa(\Sigma).$$

Thus, the theorem is proved.

According to Theorem 4, if $X'V$ is well conditioned, Φ_{Schur} is expected to be well conditioned since the singular values in Σ that are deemed too small have already been excluded after the low-rank approximation of X. However, if the underlying system has a zero eigenvalue, X' is rank deficient, which forces both Φ in DMD and Φ_{Schur} in this proposed method, to be rank deficient. The remedy to this situation is discussed in Subsection III-B.

B. Zero eigenvalue case

In so-called "standard DMD" [1], the issue of rank deficiency in $\Phi = X'V\Sigma^{-1}W$ when A has a zero eigenvalue is circumvented by constructing $\Phi = UW$, which yields not exact but rather projected eigenvectors of A (this stems from the fact that if $r < \min(m,n)$, $UU^* \neq I$.) However, this does not address the numerical instabilities that can be introduced from $\sigma_{min}(W)$ being too small. Therefore, we propose using $\Phi_{UQ} = UQ$ as the transformation matrix. Observe,

$$U\tilde{A}Q = UU^*AUQ = UQR.$$

Therefore, it yields

$$A \approx \Phi_{UQ}\, R\, \Phi_{UQ}^{+},$$

for $r < m$ and

$$A = \Phi_{UQ}\, R\, \Phi_{UQ}^{-1},$$

when $r = m$. Most importantly, the smallest possible, condition number of unity, i.e. $\kappa(\Phi_{UQ}) = 1$, is obtained rendering the guarantee on numerical stability of the proposed method.

IV. Numerical Examples

Example 1: This numerical example compares the condition numbers obtained by eDMD and Schur-based DMD in a nearly defective system. Consider the system matrix $X'X^+$ and its eigenvalues given below.

$X'X^+ =$

col. 1	col. 2	col. 3	col. 4	col. 5
3.6600	−0.7581	−1.7062	−0.9540	−0.9628
−1.6513	3.3623	−0.8971	−1.1449	−1.1537
−1.4437	−1.7414	3.4142	0.0627	−0.9461
1.5650	1.2673	1.3192	4.1664	3.0626
2.5299	2.2322	2.2841	3.0363	5.1576

$$\text{eig}(X'X^+) = \text{diag}\begin{bmatrix} 2.0958 + 0.0000i \\ 2.1293 + 0.0000i \\ 5.3121 + 0.0000i \\ 5.1117 + 0.0084i \\ 5.1117 - 0.0084i \end{bmatrix}.$$

Notice the relatively small imaginary parts of the complex conjugate eigenvalues which is the indicator that the system might be nearly defective. The near defectiveness of the system was confirmed by relatively small $\sigma_{min}(W)$ when $r \geq 4$. The results of the experiment are summarized in Table I. In the cases $r \geq 4$, the near defectiveness of the underlying system forces Φ to have very large condition numbers while the condition number of Φ_{Schur} remains small. The first state $x_1(k)$ of the discrete solutions obtained by the two methods are compared in Figure 1. In cases where $r < 4$, the low rank structures cannot capture the true nature of the underlying system and therefore, the near defectiveness of the system was not revealed in these cases yielding small condition numbers for Φ. However, the inaccuracies introduced by selecting a rank that is too low can be seen in the solution plots; see Figure 1. Therefore, for this particular example, if eDMD is applied, one will need to reckon with large condition numbers. However, with our proposed method of Schur-based DMD, selecting $r = 5$ will yield an accurate solution and even $r = 4$ can be acceptable for some applications. In all cases $\kappa(\Phi_{Schur})$ remains in single digit and reaches the bound given in Theorem 4 verifying that the bound is realistic.

TABLE I
THE EFFECT OF $\boldsymbol{\sigma_{min}(W)}$ ON $\boldsymbol{\kappa(\Phi)}$

r:	$\sigma_{min}(W)$	$\kappa(\Phi)$	$\kappa(\Phi_{Schur})$
r=2:	0.3149	4.3776	1.1965
r=3:	0.003	500.4099	1.3336
r=4:	0.0015	1.1780×10^3	3.7631
r=5:	9.6177×10^{-4}	1.8397×10^3	6.3003

Example 2: This numerical example demonstrates the numerical instabilities introduced by having a large condition number for the transformation matrix. In this numerical example consider the underlying system matrix $X'X^+$

$$X'X^+ = \begin{array}{ccc} \text{col. 1} & \text{col. 2} & \text{col. 3} \\ 1.357500 & 0.537500 & -0.537500 \\ 0.542000 & 0.353000 & 0.458000 \\ 1.004500 & -0.004500 & 0.815500 \end{array}.$$

Its eigenvalues are computed to be

$$\text{eig}(X'X^+) = \text{diag}\begin{bmatrix} 0.8950 \\ 0.8200 \\ 0.8110 \end{bmatrix}.$$

The eigenvalues are distinct but clustered together; which is common in many applications. When $r = 3$, the values of $\sigma_{min}(W) = 3.9725 \times 10^{-4}$, and $\kappa(W) = 4.3579 \times 10^3$ confirm that this system is nearly defective,

A large condition number of $\kappa(\Phi) = 4.4987 \times 10^3$ was observed. Jordan decomposition yields even higher condition number of $\kappa(\Phi_{Jordan}) = 4.7817 \times 10^3$. Now, the perturbations are introduced as follows and the solutions were plotted; see Figure 2. The first two diagonal elements of Φ are perturbed by $+0.01\%$, and $+0.08\%$ respectively. The first two diagonal elements of Φ_{Jordan} are perturbed by $+0.01\%$, and $+0.08\%$ respectively. The first two diagonal elements of Φ_{Schur} are perturbed by $+3\%$, and $+4\%$ respectively.

As shown in Figure 2, the solution produced by Schur-based DMD can withstand up to 4% perturbation in the

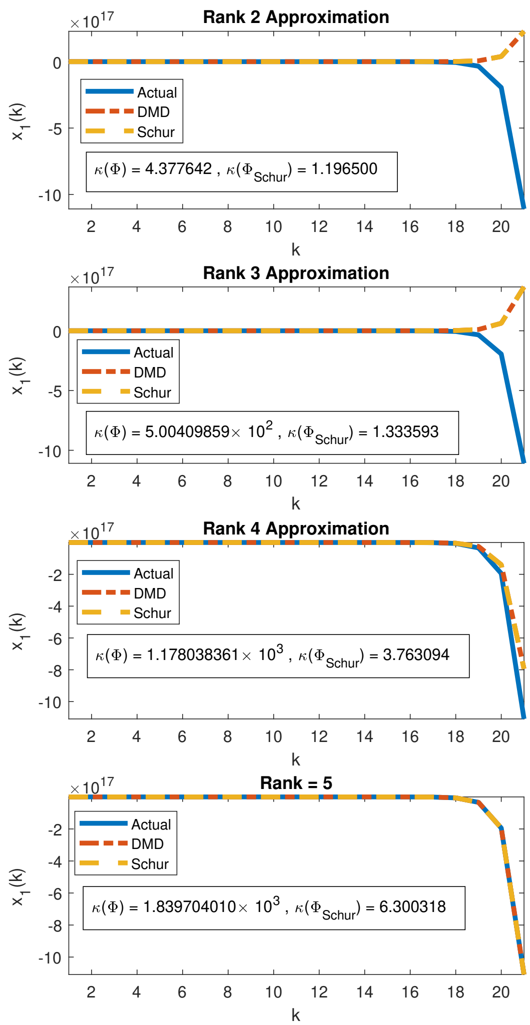

Fig. 1. The first state $\boldsymbol{x_1(k)}$, of the solution obtained by DMD and Schur DMD in Example 1.

entries of Φ_{Schur} while less than 0.1% perturbation in the entries of Φ and Φ_{Jordan} can produce erroneous solutions.

Example 3: In this numerical example consider the underlying system matrix $X'X^+$ and its eigenvalues

$X'X^+ =$

col. 1	col. 2	col. 3	col. 4	col. 5
0.4276	1.1776	0.1776	−0.8356	0.7632
0.4276	0.1776	1.1776	−0.8356	0.7632
1.4276	1.1776	1.1776	0.1644	1.7632
−3.6253	−3.8753	−3.8753	0.1644	−3.2897
2.7701	2.5201	2.5201	1.5069	1.7632

$\text{eig}(X'X^+) =$

$$\text{diag}\begin{bmatrix} 5.052799998436318 + 0.000000000000000i \\ -1.342399998436327 + 0.000000000000000i \\ 0.000004724821343 + 0.000008183603582i \\ 0.000004724821343 - 0.000008183603582i \\ -0.000009449642686 + 0.000000000000000i \end{bmatrix}.$$

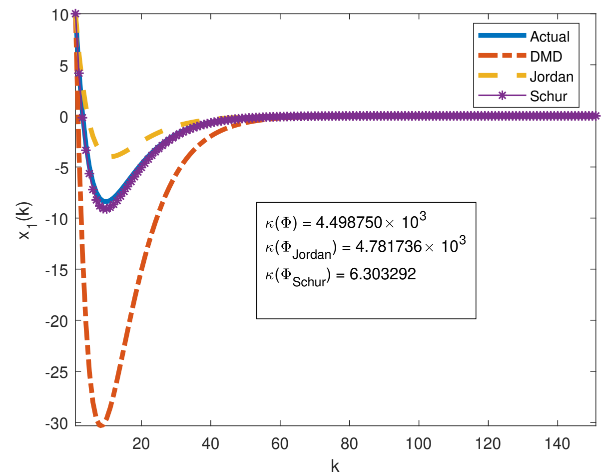

Fig. 2. The first state, $\boldsymbol{x_1(k)}$, of the solution obtained by DMD, Jordan DMD and Schur DMD in Example 2 after perturbations are introduced.

Notice the relatively small values of the last three eigenvalues, which indicate the system might have multiple zero eigenvalues. The solutions were plotted for $r = 2, 3, 4, 5$. See Figure 3. The condition number of Φ_{Jordan} was in 10^{28} range, and therefore the solution produced by Jordan decomposition is not included in the plot. The condition number of both Φ_{UW} and Φ_{UQ} remain small for $r = 2, 3, 4$. For $r = 5$,

$$\sigma_{min}(W) = 4.0667 \times 10^{-8}, \quad \kappa(W) = 4.3539 \times 10^{7}.$$

Therefore, the near defectiveness of the system is confirmed and Φ_{UW} has a large condition number of 4.3539×10^7, while $\kappa(\Phi_{UQ})$ remains unity. The perturbations are introduced as follows and the solutions are plotted; see Figure 4. The first five diagonal elements of Φ_{UW} are perturbed by $+0.06\%$, $+0.07\%$, $+0.8\%$,$+0.08\%$, $+0.08\%$ respectively. The first five diagonal elements of Φ_{UQ} are perturbed by $+9\%$, $+10\%$, $+5\%$,$+15\%$, -10% respectively. As shown in Figure 4, the solution produced by Schur-based DMD can withstand up to 15% perturbation in the entries of Φ_{Schur} while less than 1% perturbations in the entries of Φ produce erroneous solutions.

V. Conclusions

Our analysis and numerical examples show that eDMD and standard DMD can become numerically unstable when the eigendecomposition of the underlying system is ill-conditioned. Numerical examples also demonstrate that small perturbations in the transformation matrix produce erroneous results as a consequence of a large condition number. A numerically stable method which complements DMD and a realistic upper bound on the condition number of the transformation matrix for the proposed method are presented. This result is potentially consequential to computing linear approximation of Koopman operators using DMD.

References

[1] P. J. Schmid, "Dynamic mode decomposition of numerical and experimental data," *Journal of fluid mechanics*, vol. 656, pp. 5–28, 2010.

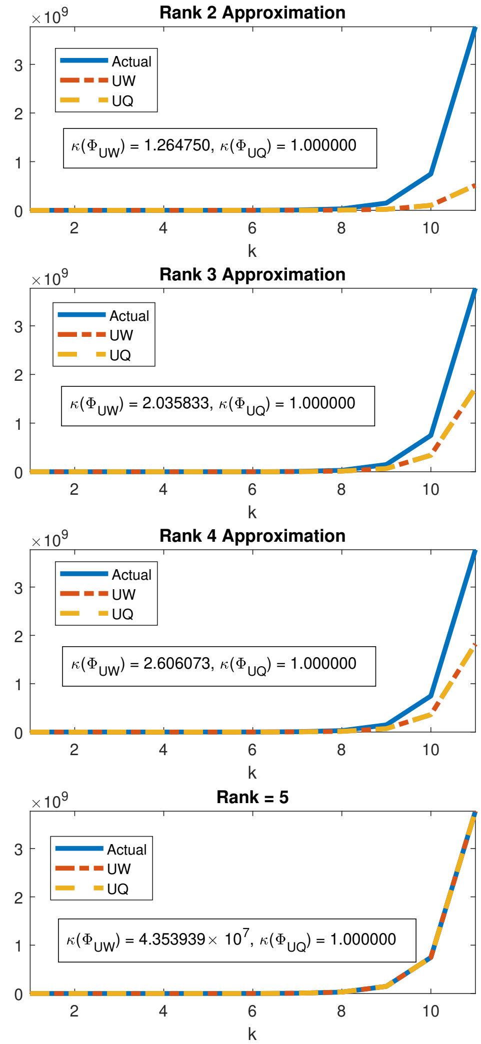

Fig. 3. The first state, $x_1(k)$, of the solution obtained by using Φ_{UW} and using Φ_{UQ} in Example 3.

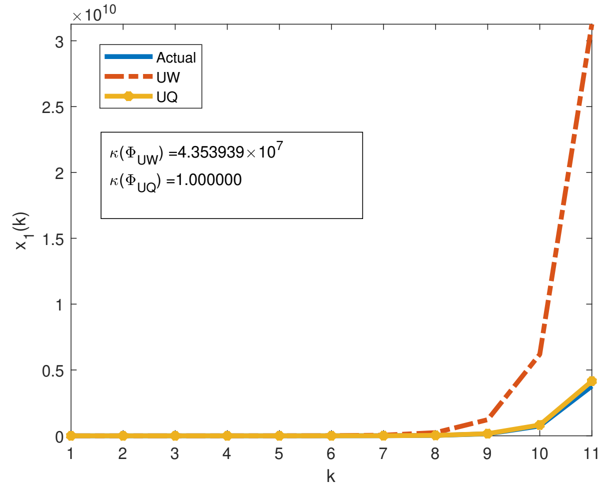

Fig. 4. The first state, $x_1(k)$, of the solution obtained by using Φ_{UW} and using Φ_{UQ} after perturbations are introduced.

[2] J. H. Tu, C. W. Rowley, D. M. Luchtenburg, S. L. Brunton, and J. N. Kutz, "On dynamic mode decomposition: Theory and applications," *arXiv preprint arXiv:1312.0041*, 2013.

[3] J. L. Proctor and P. A. Eckhoff, "Discovering dynamic patterns from infectious disease data using dynamic mode decomposition," *International health*, vol. 7, no. 2, pp. 139–145, 2015.

[4] J. A. Ramos, P. L. dos Santos, and E. I. Verriest, "A vectorized principal component approach for solving the data registration problem," in *Proceedings of the 45th IEEE Conference on Decision and Control*, 2006, pp. 1297–1302.

[5] J.-N. Juang and R. S. Pappa, "An eigensystem realization algorithm for modal parameter identification and model reduction," *Journal of Guidance, Control, and Dynamics*, vol. 8, pp. 620–627, 1985.

[6] J.-N. Juang, M. Phan, L. G. Horta, and R. W. Longman, "Identification of observer/kalman filter markov parameters - theory and experiments," *Journal of Guidance, Control, and Dynamics*, vol. 16, pp. 320–329, 1993.

[7] M. Gevers, "Identification for control: Achievements and open problems1," *IFAC Proceedings Volumes*, vol. 37, pp. 799–810, 2004.

[8] E. V. Filho and P. L. d. Santos, "A dynamic mode decomposition approach with hankel blocks to forecast multi-channel temporal series," *Control Systems Letters*, vol. 3, pp. 739–744, 2019.

[9] N. Takeishi, T. Yairi, and Y. Kawahara, "Factorially switching dynamic mode decomposition for koopman analysis of time-variant systems," in *2018 IEEE Conference on Decision and Control (CDC)*, 2018, pp. 6402–6408.

[10] P. Sashittal and D. J. Bodony, "Low-rank dynamic mode decomposition using riemannian manifold optimization," in *2018 IEEE Conference on Decision and Control (CDC)*, 2018, pp. 2265–2270.

[11] C. W. Rowley, I. Mezić, S. Bagheri, P. Schlatter, and D. S. Henningson, "Spectral analysis of nonlinear flows," *Journal of fluid mechanics*, vol. 641, pp. 115–127, 2009.

[12] N. Takeishi, Y. Kawahara, and T. Yairi, "Learning koopman invariant subspaces for dynamic mode decomposition," in *Advances in Neural Information Processing Systems*, 2017, pp. 1130–1140.

[13] M. S. Hemati, C. W. Rowley, E. A. Deem, and L. N. Cattafesta, "De-biasing the dynamic mode decomposition for applied koopman spectral analysis of noisy datasets," *Theoretical and Computational Fluid Dynamics*, vol. 31, no. 4, pp. 349–368, 2017.

[14] H. Arbabi and I. Mezic, "Ergodic theory, dynamic mode decomposition, and computation of spectral properties of the koopman operator," *SIAM Journal on Applied Dynamical Systems*, vol. 16, no. 4, pp. 2096–2126, 2017.

[15] M. Friswell, U. Prells, and S. Garvey, "Low-rank damping modifications and defective systems," *Journal of sound and vibration*, vol. 279, no. 3-5, pp. 757–774, 2005.

[16] A. Luongo, "Eigensolutions sensitivity for nonsymmetric matrices with repeated eigenvalues," *AIAA journal*, vol. 31, no. 7, pp. 1321–1328, 1993.

[17] Y. Chen, S. Chen, and Z. Liu, "Modal optimal control procedure for near defective systems," *Journal of sound and vibration*, vol. 245, no. 1, pp. 113–132, 2001.

[18] G. Yao and X. Gao, "Controllability of regular systems and defective systems with repeated eigenvalues," *Journal of Vibration and Control*, vol. 17, no. 10, pp. 1574–1581, 2011.

[19] P. Wang and X. Yang, "Eigensensitivity of damped systems with defective multiple eigenvalues," *Journal of Vibroengineering*, vol. 18, pp. 2331–2342, 2016.

[20] S. Newland, *Mechanical Vibration Analysis and Computation*. Longman Scientific and Technical, 1999.

[21] N. J. Higham, *Matrix Condition Numbers*, 1983, masters thesis, the University of Manchester.

[22] S. Demko, "Condition numbers of rectangular systems and bounds for generalized inverses," *Linear Algebra and Its Applications*, vol. 78, pp. 199–206, 1986.

[23] I. Schur, "On the characteristic roots of a linear substitution with an application to the theory of integral equations," *Math. Ann.*, vol. 66, pp. 488–510, 1909.